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COPLANAR MOTION OF TWO PLANETS, ONE HAVING A ZERO MASS.

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The supposition that two planets circulate about their central body in the same plane enables us to dispense with two differential equations of the second order in the general problem of three bodies. The further supposition, that the mass of one of them is too insignificant to have any sensible effect on the motion of the other, enables us to consider the motion of the latter as known and as taking place according to the laws of Kepler. Hence, in this case, the two co-ordinates of the planet of zero mass are the only unknowns; and they are given by two differential equations of the second order. These suppositions have, approximately, place in several cases in the solar system, but I have more especially in view the motion of the satellite Hyperion as disturbed by the action of Titan. My object in this paper is simply to point out a method of proceeding, which may, I think, be advantageously employed in this case.

Employing the usual notation x, y, r, for the rectangular co-ordinates and radius vector of the planet whose motion is to be determined, x', y', r', for the corresponding quantities belonging to the acting planet, m' the mass of the latter, and M the mass of the central body, the differential equations of motion will be

$$\frac{d^2x}{dt^2} = \frac{\partial \Omega}{\partial x}, \quad \frac{d^2y}{dt^2} = \frac{\partial \Omega}{\partial y},$$

where Ω, the potential function, has the following expression:—

$$\Omega = \frac{M}{\sqrt{(x^2+y^2)}} + m'\left[\frac{1}{\sqrt{[(x-x')^2+(y-y')^2]}} - \frac{x'x+y'y}{r'^3}\right].$$

The co-ordinates of m' satisfy the differential equations

$$\frac{d^2x'}{dt^2} + \frac{M+m'}{r'^3}x' = 0, \quad \frac{d^2y'}{dt^2} + \frac{M+m'}{r'^3}y' = 0.$$

We can, without any loss of generality, assume that the axis of x is directed toward the lower apsis of m'. Then the integrals of the last-stated differential equations are

$$x' = a'(\cos \varepsilon' - e'), \quad y' = a'\sqrt{(1-e'^2)}\sin \varepsilon',$$

where ε' is derived from the equation

$$n't + c' = \varepsilon' - e' \sin \varepsilon',$$

a', e', c' being constants, and n' being the equivalent of $\sqrt{\left\{\frac{M+m'}{a'^3}\right\}}$.

It is desirable to know what the differential equations determining x and y become when expressed in terms of any other variables. For this end Lagrange's canonical form of the equations serves very conveniently. Let the new variables be u and s, and employ the subscript $(_1)$ to denote the complete differential coefficient with respect to t of any variable to which it is attached. Then T standing for $\frac{1}{2}(x_1^2 + y_1^2)$ expressed in terms of u, s, u_1, s_1, Lagrange's canonical form of the equations is

$$\frac{d}{dt}\frac{\partial T}{\partial u_1} - \frac{\partial T}{\partial u} = \frac{\partial \Omega}{\partial u}, \quad \frac{d}{dt}\frac{\partial T}{\partial s_1} - \frac{\partial T}{\partial s} = \frac{\partial \Omega}{\partial s}.$$

As we have

$$x_1 = \frac{\partial x}{\partial u} u_1 + \frac{\partial x}{\partial s} s_1 + \frac{\partial x}{\partial t},$$

$$y_1 = \frac{\partial y}{\partial u} u_1 + \frac{\partial y}{\partial s} s_1 + \frac{\partial y}{\partial t},$$

we get

$$\begin{aligned} T = \tfrac{1}{2}\left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2\right] u_1^2 &+ \left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial s} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial s}\right) u_1 s_1 \\ + \tfrac{1}{2}\left[\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2\right] s_1^2 &+ \left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial t}\right) u_1 \\ + \left(\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial s}\frac{\partial y}{\partial t}\right) s_1 &+ \tfrac{1}{2}\left[\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2\right]. \end{aligned}$$

It is very plain from the form of Lagrange's equations that if the variables u and s were so assumed that one of them, u for instance, should disappear at once from the expressions for T and Ω, we should have an integral of the problem. For then $\frac{d}{dt}\frac{\partial T}{\partial u_1} = 0$; and, integrating, $\frac{\partial T}{\partial u_1} =$ a constant. This selection, in a theoretical sense, is always posssible, and in as many essentially distinct ways as there are first integrals of the problem, which, in the present case, are four. But, although it is easy, in innumerable ways, to make Ω depend on one variable, it is not so easy to make the six factors of the general expression for T depend solely on the same variable. And, when we inquire what equations must be satisfied for this, we find that they are essentially the same as those which are satisfied by the Eulerian multipliers. Hence, nothing is gained by approaching the problem from this side.

I propose to take u and s so that

$$x = \rho x' u + \rho y' s, \quad y = \rho y' u - \rho x' s,$$

where ρ denotes a function of t supposed known, but, for the present, left indeterminate. From these equations may be derived

$$r^2 = \rho^2 r'^2 (u^2 + s^2), \quad x'x + y'y = \rho r'^2 u.$$

Hence the potential function, in terms of u and s, becomes

$$\Omega = \frac{1}{\rho r'} \left[\frac{M}{\sqrt{(u^2 + s^2)}} + \frac{m'}{\sqrt{[(u - \rho^{-1})^2 + s^2]}} - m' \rho^2 u \right].$$

In the general expression for T we substitute the values

$$\frac{\partial x}{\partial u} = \rho x', \quad \frac{\partial y}{\partial u} = \rho y', \quad \frac{\partial x}{\partial s} = \rho y', \quad \frac{\partial y}{\partial s} = -\rho x',$$

$$\frac{\partial x}{\partial t} = \frac{d(\rho x')}{dt} u + \frac{d(\rho y')}{dt} s, \quad \frac{\partial y}{\partial t} = \frac{d(\rho y')}{dt} u - \frac{d(\rho x')}{dt} s.$$

The result is

$$T = \tfrac{1}{2} \rho^2 r'^2 (u_1^2 + s_1^2) - a'^2 n' \sqrt{(1 - e'^2)}\, \rho^2 (u s_1 - s u_1)$$
$$+ \tfrac{1}{2} \left[a'^2 n'^2 \left\{ \frac{2a'}{r'} - 1 \right\} \rho^2 + 2r' \frac{dr'}{dt} \rho \frac{d\rho}{dt} + r'^2 \frac{d\rho^2}{dt^2} \right] (u^2 + s^2).$$

For the sake of brevity we may write, h_1, h_2, h_3 being known functions of t,

$$T = \tfrac{1}{2} h_1 (u_1^2 + s_1^2) - h_2 (u s_1 - s u_1) + \tfrac{1}{2} h_3 (u^2 + s^2).$$

This, substituted in Lagrange's canonical form of the differential equations, gives as the equations of the problem,

$$\frac{d}{dt} \left(h_1 \frac{du}{dt} \right) + 2h_2 \frac{ds}{dt} - h_3 u + \frac{dh_2}{dt} s = \frac{\partial \Omega}{\partial u},$$

$$\frac{d}{dt} \left(h_1 \frac{ds}{dt} \right) - 2h_2 \frac{du}{dt} - \frac{dh_2}{dt} u - h_3 s = \frac{\partial \Omega}{\partial s}.$$

Let us now adopt a more general independent variable than the time. Calling this ζ, let $dt = \theta d\zeta$, in which θ may be regarded as a function either of t or ζ. The second supposition will be the more advantageous. In either case as we obtain, on integrating, u and s as functions of ζ, it will be necessary to have the values of ζ which correspond to given values of the time, and thus the inverse function will have to be considered. Then, in terms of the new independent variable,

$$\frac{d}{d\zeta} \left(\frac{h_1}{\theta} \frac{du}{d\zeta} \right) + 2h_2 \frac{ds}{d\zeta} - \theta h_3 u + \frac{dh_2}{d\zeta} s = \frac{\partial (\theta \Omega)}{\partial u},$$

$$\frac{d}{d\zeta} \left(\frac{h_1}{\theta} \frac{ds}{d\zeta} \right) - 2h_2 \frac{du}{d\zeta} - \frac{dh_2}{d\zeta} s - \theta h_3 u = \frac{\partial (\theta \Omega)}{\partial s}.$$

We can now consider how ρ and θ should be assumed in order that the differential equations may be most simplified. In the first place it appears important that the potential function Ω should be freed from the independent variable ζ. This is accomplished by putting $\rho = 1$. In the second place it seems we cannot readily do better than take the eccentric anomaly ε' of the attracting planet as the independent variable ζ. Then

$$dt = \frac{r'}{a'n'}\,d\varepsilon', \quad \text{and} \quad \theta = \frac{r'}{a'n'}\,.$$

Also we have

$$h_1 / \theta = a'^2 n' (1 - e' \cos \varepsilon'), \quad h_2 = a'^2 n' \sqrt{(1 - e'^2)}, \quad \theta h_3 = a'^2 n' (1 + e' \cos \varepsilon'),$$

$$\frac{\theta}{\rho r'} = \frac{1}{a'n'}\,, \quad M + m' = a'^3 n'^2.$$

For the sake of simplicity let the signification of Ω be changed, and, putting $\dfrac{m'}{M + m'} = \nu$, let

$$\Omega = \frac{1 - \nu}{\sqrt{(u^2 + s^2)}} + \frac{\nu}{\sqrt{[(u - 1)^2 + s^2]}} - \nu u.$$

Then our differential equations take the form

$$\frac{d}{d\varepsilon'}\left[(1 - e' \cos \varepsilon')\frac{du}{d\varepsilon'}\right] + 2\sqrt{(1 - e'^2)}\frac{ds}{d\varepsilon'} - (1 + e' \cos \varepsilon')\, u = \frac{\partial \Omega}{\partial u}\,,$$

$$\frac{d}{d\varepsilon'}\left[(1 - e' \cos \varepsilon')\frac{ds}{d\varepsilon'}\right] - 2\sqrt{(1 - e'^2)}\frac{du}{d\varepsilon'} - (1 + e' \cos \varepsilon')\, s = \frac{\partial \Omega}{\partial s}\,.$$

It will be noticed that the potential function Ω is, by this assumption of variables, completely freed from co-ordinates expressing the position of the attracting planet; and that the two factors $1 - e' \cos \varepsilon'$ and $1 + e' \cos \varepsilon'$, very simple functions of the independent variable ε', are the only evidences of the position of this body in the differential equations. And, of the four elements of its orbit, e' is the only one we have to deal with.

We propose now to see whether the introduction of elliptic co-ordinates will bring about any simplification in the problem. Supposing

$$x_1 = s, \quad x_2 = u - \tfrac{1}{2},$$

let

$$\frac{x_1^2}{a_1 + \lambda_1} + \frac{x_2^2}{a_2 + \lambda_1} = 1, \quad \text{and} \quad \frac{x_1^2}{a_1 + \lambda_2} + \frac{x_2^2}{a_2 + \lambda_2} = 1,$$

be the equations of a confocal ellipse and hyperbola, a_1 and a_2 being constants, and λ_1 and λ_2 the new variables destined to take the place of u and s. By eliminating x_2^2 from these equations we obtain

$$\frac{a_1 - a_2}{(a_1 + \lambda_1)(a_1 + \lambda_2)}\, x_1^2 = 1\,;$$

whence
$$x_1 = \sqrt{\left[\frac{(a_1+\lambda_1)(a_1+\lambda_2)}{a_1-a_2}\right]}.$$

The expression of x_2 in terms of λ_1 and λ_2 is obtained from this by simply interchanging a_1 and a_2. Thus
$$x_2 = \sqrt{\left[\frac{(a_2+\lambda_1)(a_2+\lambda_2)}{a_2-a_1}\right]}.$$

We now proceed to find what Ω becomes in terms of λ_1 and λ_2. By taking the sum of the squares of the last two equations we get

$$x_1^2 + x_2^2 = a_1 + a_2 + \lambda_1 + \lambda_2.$$

Thus far a_1 and a_2 have been left indeterminate, but we now assume

$$a_2 - a_1 = \tfrac{1}{4}.$$

Then
$$\begin{aligned}
u^2 + s^2 &= (x_2 + \tfrac{1}{2})^2 + x_1^2 \\
&= 2a_2 + \lambda_1 + \lambda_2 + 2\sqrt{[(a_2+\lambda_1)(a_2+\lambda_2)]} \\
&= [\sqrt{(a_2+\lambda_1)} + \sqrt{(a_2+\lambda_2)}]^2, \\
\sqrt{(u^2+s^2)} &= \sqrt{(a_2+\lambda_1)} + \sqrt{(a_2+\lambda_2)}, \\
(u-1)^2 + s^2 &= (x_2 - \tfrac{1}{2})^2 + x_1^2 \\
&= 2a_2 + \lambda_1 + \lambda_2 - 2\sqrt{[(a_2+\lambda_1)(a_2+\lambda_2)]}, \\
\sqrt{[(u-1)^2+s^2]} &= \sqrt{(a_2+\lambda_1)} - \sqrt{(a_2+\lambda_2)}, \\
u &= 2\sqrt{[(a_2+\lambda_1)(a_2+\lambda_2)]} + \tfrac{1}{2}.
\end{aligned}$$

For the sake of brevity we will now put

$$\sqrt{(a_2+\lambda_1)} = p, \quad \sqrt{(a_2+\lambda_2)} = q.$$

Then it is plain Ω may be written

$$\begin{aligned}
\Omega &= \frac{1-\nu}{p+q} + \frac{\nu}{p-q} - 2\nu pq \\
&= \frac{1-\nu}{p+q} + \frac{\nu}{p-q} - \tfrac{1}{2}\nu(p+q)^2 + \tfrac{1}{2}\nu(p-q)^2.
\end{aligned}$$

We have now to deal with T. By taking the logarithms of the values of x_1^2 and x_2^2, and then differentiating, we obtain

$$2\frac{dx_1}{x_1} = \frac{d\lambda_1}{a_1+\lambda_1} + \frac{d\lambda_2}{a_1+\lambda_2},$$

$$2\frac{dx_2}{x_2} = \frac{d\lambda_1}{a_2+\lambda_1} + \frac{d\lambda_2}{a_2+\lambda_2}.$$

Whence may be derived

$$4(dx_1^2+dx_2^2)=\left[\frac{x_1^2}{(a_1+\lambda_1)^2}+\frac{x_2^2}{(a_2+\lambda_1)^2}\right]d\lambda_1^2+\left[\frac{x_1^2}{(a_1+\lambda_2)^2}+\frac{x_2^2}{(a_2+\lambda_2)^2}\right]d\lambda_2^2$$
$$+\,2\left[\frac{x_1^2}{(a_1+\lambda_1)(a_1+\lambda_2)}+\frac{x_2^2}{(a_2+\lambda_1)(a_2+\lambda_2)}\right]d\lambda_1 d\lambda_2.$$

On substituting in the factor of $d\lambda_1\,d\lambda_2$ the values of x_1^2 and x_2^2 it vanishes, and the expression takes the form

$$4(dx_1^2+dx_2^2)=\frac{\lambda_1-\lambda_2}{(a_1+\lambda_1)(a_2+\lambda_1)}\,d\lambda_1^2+\frac{\lambda_2-\lambda_1}{(a_1+\lambda_2)(a_2+\lambda_2)}\,d\lambda_2^2.$$

Or, in terms of p and q, we have

$$du^2+ds^2=\frac{p^2-q^2}{p^2-\frac{1}{4}}\,dp^2+\frac{q^2-p^2}{q^2-\frac{1}{4}}\,dq^2.$$

In like manner we get

$$uds-sdu=(p+q)\left[\sqrt{\left(\frac{\frac{1}{4}-q^2}{p^2-\frac{1}{4}}\right)}\,dp-\sqrt{\left(\frac{p^2-\frac{1}{4}}{\frac{1}{4}-q^2}\right)}\,dq\right].$$

The former expression for T was

$$T=\tfrac{1}{2}(1-e'\cos\varepsilon')\frac{du^2+ds^2}{d\varepsilon'^2}-\sqrt{(1-e'^2)}\,\frac{uds-sdu}{d\varepsilon'}+\tfrac{1}{2}(1+e'\cos\varepsilon')(u^2+s^2);$$

hence, if we abbreviate by putting

$$\sqrt{\left(\frac{\frac{1}{4}-q^2}{p^2-\frac{1}{4}}\right)}=a,$$

$$T=\tfrac{1}{2}(1-e'\cos\varepsilon')\left[(1+a^2)\frac{dp^2}{d\varepsilon'^2}+\left(1+\frac{1}{a^2}\right)\frac{dq^2}{d\varepsilon'^2}\right]$$
$$-\sqrt{(1-e'^2)}\,(p+q)\left[a\frac{dp}{d\varepsilon'}-\frac{1}{a}\frac{dq}{d\varepsilon'}\right]+\tfrac{1}{2}(1+e'\cos\varepsilon')(p+q)^2.$$

T and Ω are somewhat simplified if we adopt variables ρ and σ, such that

$$p+q=\rho,\quad p-q=\sigma.$$

Also, for the sake of brevity, put

$$\tfrac{1}{2}\left(a+\frac{1}{a}\right)=h,\quad \tfrac{1}{2}\left(a-\frac{1}{a}\right)=k.$$

Then we have

$$T=\tfrac{1}{2}(1-e'\cos\varepsilon')\left[h^2\left(\frac{d\rho^2}{d\varepsilon'^2}+\frac{d\sigma^2}{d\varepsilon'^2}\right)-2hk\frac{d\rho}{d\varepsilon'}\frac{d\sigma}{d\varepsilon'}\right]$$
$$-\sqrt{(1-e'^2)}\,\rho\left(k\frac{d\rho}{d\varepsilon'}+h\frac{d\sigma}{d\varepsilon'}\right)+\tfrac{1}{2}(1+e'\cos\varepsilon')\,\rho^2,$$

$$\Omega=\frac{1-\nu}{\rho}+\frac{\nu}{\sigma}-\tfrac{1}{2}\nu\rho^2+\tfrac{1}{2}\nu\sigma^2.$$

By this transformation Ω is considerably simplified; but, as more than offsetting this, T is rendered complex. As the expression for α in terms of these variables is

$$\alpha = \sqrt{\left[\frac{1 - (\rho - \sigma)^2}{(\rho + \sigma)^2 - 1}\right]},$$

it will be perceived that h and k are trigonometrical functions of the angles of the triangle whose sides are 1, ρ, and σ, which might have been anticipated from geometrical considerations. Thus it appears no advantage would result from the employment of elliptic co-ordinates.

Returning, therefore, to the quasi-rectangular co-ordinates u and s, it seems some advantage would be gained if we adopt a new system of co-ordinates u and s, such that the new system is expressed, in terms of the old, as follows: —

$$u = u + s\sqrt{(-1)}, \quad s = u - s\sqrt{(-1)}.$$

We can also adopt the trigonometrical exponential corresponding to the arc ε' as the independent variable. Calling this $\zeta = e^{\varepsilon' \sqrt{(-1)}}$, an operator D is adopted, equivalent to $\zeta \dfrac{d}{d\zeta}$, so that $D \,.\, \zeta^i = i\zeta^i$.

In terms of the new variables, Ω has the expression

$$\Omega = \frac{1 - \nu}{\sqrt{(us)}} + \frac{\nu}{\sqrt{[(u - 1)(s - 1)]}} - \tfrac{1}{2}\nu\,(u + s).$$

And the differential equations are

$$D\,\{[1 - \tfrac{1}{2}e'(\zeta + \zeta^{-1})]\,Du\} + 2\sqrt{(1 - e'^2)}\,Du + [1 + \tfrac{1}{2}c'(\zeta + \zeta^{-1})]\,u = -2\frac{\partial\Omega}{\partial s},$$

$$D\,\{[1 - \tfrac{1}{2}e'(\zeta + \zeta^{-1})]\,Ds\} - 2\sqrt{(1 - e'^2)}\,Ds + [1 + \tfrac{1}{2}e'(\zeta + \zeta^{-1})]\,s = -2\frac{\partial\Omega}{\partial u}.$$

Only one of these equations need be actually employed, as either can be obtained from the other by changing the sign of $\sqrt{(-1)}$. We have

$$-2\frac{\partial\Omega}{\partial s} = \frac{1 - \nu}{\sqrt{u}\,.\,\sqrt{s^3}} + \frac{\nu}{\sqrt{(u - 1)}\,.\,\sqrt{(s - 1)^3}} + \nu,$$

$$-2\frac{\partial\Omega}{\partial u} = \frac{1 - \nu}{\sqrt{u^3}\,.\,\sqrt{s}} + \frac{\nu}{\sqrt{(u - 1)^3}\,.\,\sqrt{(s - 1)}} + \nu.$$

For the purpose of integrating these equations, we may adopt the method of indeterminate coefficients; and we may employ, as proper to represent the values of u and s, the infinite series

$$u = \Sigma' \,.\, \mathbf{a}_{i,j,k}\, \zeta^{ic + jc' + k},$$

$$s = \Sigma' \,.\, \mathbf{a}_{i,j,k}\, \zeta^{-ic - jc' - k}.$$

Here i, j, and k denote positive or negative integers, zero included; and the summation must be extended so as to include all values for i, j, or k from $-\infty$ to $+\infty$. The $\mathfrak{a}$ and c, c' are constants and functions of the four quantities e', ν, a and e; a and e being two of the four arbitrary constants introduced by integration. The two remaining arbitrary constants serve only to complete the two elementary arguments which belong to the attracted planet, and, in this method of integration, they can pass unnoticed.

If we suppose that the orbit of the attracting planet is circular, the differential equations reduce to the very simple form

$$(D + 1)^2 u = -2 \frac{\partial \Omega}{\partial s},$$

$$(D - 1)^2 s = -2 \frac{\partial \Omega}{\partial u}.$$

And, in this case, an integral can be found. For multiplying the first by Ds, and the second by Du, the sum of the equations, thus multiplied, is an exact derivative. Integrating, we get

$$Du\, Ds + us + 2\Omega = 2C,$$

C being the arbitrary constant.

This integral equation may be combined with the differential equations in such a way that one of the terms, regarded as the most difficult of expression in a developed form, may be eliminated. For example, if this is taken to be the term $\frac{\nu}{\sqrt{[(u-1)(s-1)]}}$ of Ω, the equations serving to determine the $\mathfrak{a}$ may be taken to be

$$(s - 1) D (D + 2) u + \tfrac{1}{2} Du\, Ds + \frac{1 - \nu}{\sqrt{(us)}^3} u + \tfrac{3}{2} (u - \nu)(s - \nu) + C = 0,$$

$$(u - 1) D (D - 2) s + \tfrac{1}{2} Du\, Ds + \frac{1 - \nu}{\sqrt{(us)}^3} s + \tfrac{3}{2} (u - \nu)(s - \nu) + C = 0,$$

in which the constant C is not identical with the former C. One of these equations suffices, as the other is a consequence of it. The difference of these equations is simpler than either of them, and may be of use. It is

$$D\,[(u - 1)\, Ds - (s - 1)\, Du - 2\,(u - 1)(s - 1)] = \frac{1 - \nu}{\sqrt{(us)}^3}(u - s).$$

In attempting to derive periodic series for the co-ordinates of Hyperion, it appears to me that it will be easier, in the first instance, to assume that Titan describes a circular orbit. And, in the next place, to assume that the perturbations are periodic functions of the mean elongation of the two bodies. And, as it may very easily happen that the terms, depending on the second and higher powers of the disturbing force, may quite alter the values of the coefficients, it will be well to employ the method of mechanical quadratures. Starting Hyperion from its line of conjunction with Titan, and at right angles to this line, with an assumed velocity, trace out its path until the elongation, between the two bodies, amounts to 180°. Then, if Hyperion is again moving at right angles to its radius vector, the velocity at the start has been rightly assumed. But if not, one makes another trial; and, by interpolating between the two results, a velocity is obtained which will more nearly bring about this condition. And continued repetition of these trials will enable us to discover, with all desired approximation, the velocity which fulfils this condition. When the path of Hyperion, corresponding to this velocity, has been traced out, it will be easy, by the well-known processes of mechanical quadratures, to assign the periodic series representing the co-ordinates of the satellite under the supposed conditions.

When this is done, corrections to the co-ordinates, proportional to the first power of the satellite's proper eccentricity, can be obtained by the integration of a linear differential equation. By comparison of these with observation an approximate value of this proper eccentricity will be obtained; a thing to be desired as we seem to know next to nothing about it at present. Also one will be enabled to decide whether the motion of the mean anomaly is more rapid than that of the mean longitude, as has been asserted, without sufficient reason as it seems to me.

As illustrating this point, suppose that our moon, instead of having an eccentricity about 0.055, had one about 0.001. Then the variation would be the prevailing inequality, and the moon would appear to be in perigee always about syzygies, and in apogee about quadratures. In consequence the perigee would appear to retrograde with reference to the sun as fast as the moon advances with reference to the same body. And yet the relation between the motion of the argument, denominated the mean anomaly, and the motion of the mean longitude, would be nearly the same as it is at present. But the position of the perisaturnium of Hyperion has been concluded from its observed shortest and longest *radii vectores.* This is allowable only when the inequality, called the equation of the centre, is the overpowering one.

After the terms, proportional to the first power of the eccentricity, have been obtained, those factored by the second, third, etc., powers, can be derived by integrating differential equations of the same character.

In applying the process of mechanical quadratures to the motion of Hyperion, one will meet the difficulty of the uncertain value of the mass of Titan. But this cannot be avoided; an assumption must be made, and the results afterwards corrected by comparison with observation.